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Modal Synthesis When Modeling Damping by Use of Fractional Derivatives

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The fractional derivative damping model and some of its properties are discussed. Small problems involving fractional derivatives may be solved in the Fourier domain. Here modal synthesis is used to solve the equations of motion in the time domain. A state-space vector containing fractional derivatives of the displacements is then introduced, and the system of equations is expanded. The resulting system of equations is decoupled by use of the complex eigenvectors of the system. The solutions of the decoupled equations are superposed, weighted by the eigenmodes, to give the response of the structure. The computational effort can be reduced by taking into account some of the properties of the system.

Introduction

DAMPING is always present in the motion of structures. Therefore, to fully model the dynamic behavior of a structure, its damping properties have to be taken into account as well as its stiffness and mass. There are several ways to model damping, each having its advantages and disadvantages. One common and simple way is to use the viscous model, i.e., the damping force is taken as proportional to the first time derivative of the displacement. This means that the damping is proportional to frequency, which results in very high damping at high frequencies. By adding terms containing higher order integer derivatives of displacement and force, the force–displacement relationship may be better fitted to experimental data. However, often many terms have to be used to get a good fit.[1]

Another commonly used model is the hysteretic model. In that model the ordinary modulus of elasticity is replaced by a complex-valued modulus, implying that the damping becomes independent of frequency. This model works well for harmonic vibration, but it leads to noncausal responses to transient loads.[2] A third linear model, the fractional derivative model as studied in this paper, results in moderate damping at high frequencies and leads to causal responses to transient loads.

Already in 1936, Gemant[3] observed that the harmonic motion of viscoelastic bodies behaves like frequency raised to fractional powers and suggested the use of fractional derivatives. In 1979, Bagley and Torvik[4] studied the stresses resulting from sinusoidal strain in the fractional derivative damping model. These stresses were shown to be sinusoidal when the initial transients had died out and the hysteresis loop was found to be elliptical. Bagley and Torvik[5] have also developed finite element equations for the fractional derivative model in the Laplace domain. Finite element equations in the time domain were studied by Bagley and Calico,[6] who rewrote the equations to make it possible to solve problems with nonhomogeneous initial conditions. Computational algorithms for finite element simulations have been derived by Padovan.[7]

Makris and Constantinou[8] used a model including fractional derivatives to describe single-degree-of-freedom dampers used for vibration and seismic isolation. The responses of the dampers were calculated by use of the discrete Fourier transform. Gaul et al.[9] studied a way to calculate the impulse response function of a damped oscillator. Both a fractional derivative damping model and a hysteretic damping model were used. The authors also commented on the causal behavior of the fractional derivative damping model. The fractional derivative model in convolution integral form is studied by Enelund and Olsson.[10]

When equations of motion contain fractional derivatives, it is often convenient to transform them into, and solve them within, the Fourier domain. One reason for the present investigation has been the possibility to use the fractional derivative model for polymer railway pads in an already existing railway model. This model is described by its modal components, and therefore a modal synthesis method to solve fractional equations of motion in the time domain is studied.

Fractional Derivative Damping Model

In the most general case the relationship between uniaxial stress σ and strain ϵ in the fractional derivative model may be expressed as

$$\sigma + \sum_{m=1}^{M} l_m \frac{\mathrm{d}^{\beta_m}\sigma}{\mathrm{d}t^{\beta_m}} = E_0\epsilon + \sum_{n=1}^{N} E_n \frac{\mathrm{d}^{\alpha_n}\epsilon}{\mathrm{d}t^{\alpha_n}} \tag{1}$$

Here the factors l_m, E_0, and E_n together with the fractional orders α_n and β_m of the derivatives are material parameters. For integer order derivatives in Eq. (1), the standard viscoelastic model is recovered. The fractional time derivative of order α, $0 < \alpha < 1$, is often defined as[11]

$${}_0D_t^{\alpha}[x(t)] = \frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{x(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau \tag{2}$$

where Γ is the gamma function. One property of the fractional derivative is that if a function is zero for negative times, then the Fourier transform of the fractional derivative of the function equals $(i\omega)^\alpha$ times the Fourier transform of the function itself.[6] Denoting the Fourier transform of [ ], by [^], one may write

$$\left[\widehat{\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}}\right] = (i\omega)^\alpha[\widehat{x(t)}] \tag{3}$$

where $(i\omega)^\alpha$ is the principal root and the Fourier transform is defined as

$$[\widehat{x(t)}] = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,\mathrm{d}t \tag{4}$$

It has been observed[5] that only five parameters often suffice in the stress–strain relationship (1) to get a good fit to experimental data,

$$\sigma + l\frac{\mathrm{d}^\beta\sigma}{\mathrm{d}t^\beta} = E_0\epsilon + E_1\frac{\mathrm{d}^\alpha\epsilon}{\mathrm{d}t^\alpha} \tag{5}$$

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To obtain a thermodynamically well-behaved model some constraints have to be placed on the parameters of the model. Bagley and Torvik[12] state that these constraints are

$$E_0 \geq 0, \qquad E_1 > 0, \qquad l > 0, \qquad E_1/l \geq E_0, \qquad \alpha = \beta \tag{6}$$

In other words, the orders of the two fractional derivatives in the model have to be equal, which means that the original five-parameter model reduces to a four-parameter model. If l is set to zero, the damping will be large at high frequencies (as for the viscous damper). With l and E_1 equal to zero, Hooke's law is obtained. Note that $E_1/l = E_0$ is equivalent to zero damping.

One-Degree-of-Freedom System

As an example, the one-degree-of-freedom system in Fig. 1 is studied. It consists of a mass m, $m = 1$ kg, and a damped spring. The latter is modeled by use of the relationship in Eq. (5),

$$f + l\frac{\mathrm{d}^\beta f}{\mathrm{d}t^\beta} = kx + c\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} \tag{7}$$

Here f is the force in the spring. A unit impulse load is applied to the mass at time $t = t_1$, and the resulting displacement $x(t)$ of the mass is calculated. Only homogeneous initial conditions are considered, i.e., the system is at rest at times $t \leq 0$.

Newton's second law applied to the mass gives

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = F - f \tag{8}$$

Here F is the applied load. By taking the Fourier transform of Eqs. (7) and (8) and eliminating the force $\hat{f}$, one has

$$\hat{x} = \frac{1}{(-m\omega^2 + \{[c(i\omega)^\alpha + k]/[l(i\omega)^\beta + 1]\})}\hat{F} \tag{9}$$

In the examples studied the load is a unit impulse load at time $t = t_1$, i.e., $F = \tilde{F}\delta(t - t_1)$ where $\tilde{F} = 1\mathrm{N}$, which has the Fourier transform $\hat{F} = \tilde{F}e^{-i\omega t_1}$. The displacement may be transformed back into the time domain numerically by use of the discrete Fourier transform.

In Fig. 2 the response of the one-degree-of-freedom system in Fig. 1 is plotted when the order $\alpha(=\beta)$ of the derivatives is set to 0.50, 0.67, and 1.0, respectively. The other spring parameters are given in Fig. 2. As may be seen, the motion decays faster when the order of the derivative is high. This also reduces the frequency of the motion.

Another way of treating Eqs. (7) and (8) is to eliminate the force f in the spring directly, by taking the β derivative of Eq. (8). As the mass is at rest at time $t = 0$, the composition rule yields[11]

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right) = \frac{\mathrm{d}^{2+\beta}x}{\mathrm{d}t^{2+\beta}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\mathrm{d}^\beta x}{\mathrm{d}t^\beta}\right) \tag{10}$$

Equations (7) and (8) now combine to the differential equation

$$lm\frac{\mathrm{d}^{2+\beta}x}{\mathrm{d}t^{2+\beta}} + m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + c\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} + kx = l\frac{\mathrm{d}^\beta F}{\mathrm{d}t^\beta} + F \tag{11}$$

This differential equation is of higher order than the original equations and it needs more initial conditions. Five initial conditions are

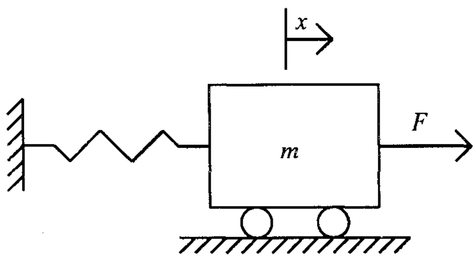

Fig. 1 One-degree-of-freedom system.

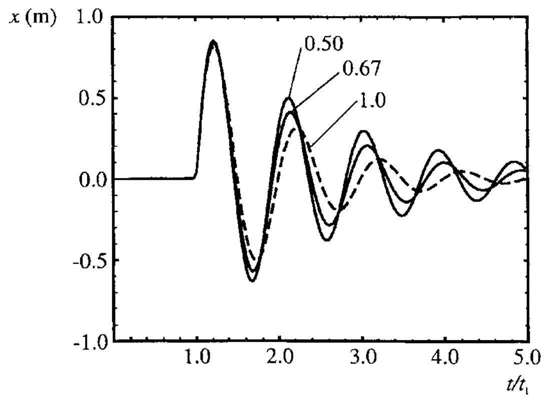

Fig. 2 Calculated response of one-degree-of-freedom system in Fig. 1 to unit impulse load at $t/t_1 = 1$. Order of derivatives $\alpha(=\beta)$ is 0.50, 0.67, and 1.0, respectively. Parameters are $m = 1$ kg, $k = 1.0$ N/m, $c = 0.4$ Ns$^\alpha$/m, and $l = 0.1$ s$^\alpha$.

needed with Eq. (11) if $\alpha = \beta$, whereas Eqs. (7) and (8) together only have three initial conditions. Also when using the standard viscoelastic model where α and β are taken as 1 (unity), an additional initial condition occurs. Flügge[13] discusses these initial conditions and indicates that they correspond to different internal states of the material. Here all initial conditions are assumed to be zero, which will cause problems if the load is infinite at time $t = 0$, e.g., an impulse load. The problems might be avoided by letting the lower limit in the derivative definition (2) be 0^-.

The operator $\mathrm{d}^\beta/\mathrm{d}t^\beta$ has an eigenfunction, i.e., a function that makes the right-hand side of Eq. (11) zero. Oldham and Spanier[11] give the eigenfunction as

$$F(t) = t^{\beta-1}\sum_{j=0}^{\infty}\frac{C^j t^{j\beta}}{\Gamma[(j+1)\beta]} \tag{12}$$

When the constant C is $C = -1/l$, Eq. (11) will be homogeneous.

It may be noted that taking the Fourier transform of Eq. (11), one obtains exactly Eq. (9).

Modal Synthesis

The Fourier transformation method may be extended to systems with several degrees of freedom, but the method might be time consuming. A modal analysis approach can be more advantageous.

A discrete system consisting of masses and damped springs is considered. The springs are modeled by relationship (5),

$$c_i\frac{\mathrm{d}^{\alpha_i}\Delta_i}{\mathrm{d}t^{\alpha_i}} + k_i\Delta_i = l_i\frac{\mathrm{d}^{\alpha_i}f_i}{\mathrm{d}t^{\alpha_i}} + f_i \tag{13}$$

where f_i is the force in the ith spring and Δ_i the extension of spring i. Equations (13) are to be solved together with Newton's second law for the masses to obtain the displacements of the masses. Here it will be assumed that all of the parameters l_i have the same value l and that the orders of the derivative of the force in the spring equations are all the same, i.e., all $\alpha_i = \alpha$. Again only homogeneous initial conditions will be considered. The forces f_i in the springs may be eliminated from the system of differential equations, and the equations of motion of a discrete (or discretized) N-degree-of-freedom structure can be written as, compare with Eq. (11),

$$lM\frac{\mathrm{d}^{2+\alpha}x}{\mathrm{d}t^{2+\alpha}} + M\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + C\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} + Kx = F + l\frac{\mathrm{d}^\alpha F}{\mathrm{d}t^\alpha} \tag{14}$$

Here M, C, and K are $N \times N$ matrices; x is the displacement vector; and F is the vector of applied forces.

In many problems, the order α of the derivative is (or can be approximated as) a rational number $\alpha = m/n$, where m and n are integers. By introducing a state-space vector (T for transpose)

$$y = \left(x \;\; \frac{\mathrm{d}^{1/n}x}{\mathrm{d}t^{1/n}} \;\; \cdots \;\; \frac{\mathrm{d}x}{\mathrm{d}t} \;\; \cdots \;\; \frac{\mathrm{d}^{(2n+m-1)/n}}{\mathrm{d}t^{(2n+m-1)/n}}\right)^T \tag{15}$$

with length $(2n+m)N$, and adding a set of dummy equations, Eq. (14) may be rewritten as (if the parameter l is zero, only the first $2nN$ elements in the vector should be used)

$$A\frac{\mathrm{d}^{1/n}y}{\mathrm{d}t^{1/n}} + By = R \tag{16}$$

The right-hand side vector R contains the applied forces and their derivatives,

$$R = \begin{pmatrix} F + l\dfrac{\mathrm{d}^{\alpha}F}{\mathrm{d}t^{\alpha}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{17}$$

The $(2n+m)N \times (2n+m)N$ matrices A and B are described in the Appendix. The first N rows in these matrices make up the system of equations (14) and the following equations in system (16) are identities.

Decoupling

Equations (16) can be decoupled by use of the eigenvectors $\tilde{\rho}$ of the system. These are collected as the columns of the $(2n+m)N \times (2n+m)N$ matrix P (capital rho). Modal displacements $q(t)$ such that $Pq = y$ are introduced in Eq. (16). By premultiplying the equation by P^T, and by making use of the orthogonality properties of the eigenvectors, Eq. (16) becomes fully decoupled. One obtains

$$\mathrm{diag}(a_i)\frac{\mathrm{d}^{1/n}q}{\mathrm{d}t^{1/n}} + \mathrm{diag}(b_i)q = Q \tag{18}$$

where $\mathrm{diag}(a_i)$ and $\mathrm{diag}(b_i)$ denote the diagonal matrices P^TAP and P^TBP, respectively, and $Q = P^TR$. Distinct eigenvalues have been presupposed.

According to Bagley and Calico[6] a particular solution of each modal equation in Eq. (18) may be written as an integral,

$$q_i(t) = \int_0^t \frac{\mathrm{d}^{1-(1/n)}}{\mathrm{d}t^{1-(1/n)}}\left[E_{1/n}\left(-\frac{b_i}{a_i}\tau^{1/n}\right)\right]\frac{1}{a_i}Q_i(t-\tau)\,\mathrm{d}\tau \tag{19}$$

where E_β is the β-order Mittag–Leffler function[14] defined as

$$E_\beta(x) = \sum_{p=0}^{\infty}\frac{x^p}{\Gamma(1+p\beta)} \tag{20}$$

There are homogeneous solutions to Eq. (18), but as the initial conditions also are homogeneous, these solutions are not needed. The differentiation inside the integral in solution (19) may be executed term by term. The expression for the derivative of each term is found in Oldham and Spanier.[11] The resulting sum is

$$\frac{\mathrm{d}^{(n-1)/n}}{\mathrm{d}t^{(n-1)/n}}\left[E_{1/n}\left(-\frac{b_i}{a_i}t^{1/n}\right)\right] = \sum_{p=0}^{\infty}\frac{(-b_i/a_i)^p t^{(p-n+1)/n}}{\Gamma[(p+1)/n]} \tag{21}$$

Once the modal displacements q_i have been calculated, the state-space vector can be obtained by superposition of the modal displacements according to $y = Pq$. The physical displacement x is found as the first N elements of the state-space vector y.

Numerical Example 1

As a numerical example, the two-degree-of-freedom system in Fig. 3 is studied. The order of the derivatives in the spring relations is approximated to $\alpha_i = m/n = \frac{2}{3}$, and for numerical simplicity, the parameters l_i are taken to be zero. The remaining numerical values are given beneath the figure. The external load is a unit impulse at time $t = t_1$ applied to mass m_1.

The system of equations to be solved is

$$M\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + C\frac{\mathrm{d}^{\frac{2}{3}}x}{\mathrm{d}t^{\frac{2}{3}}} + Kx = F \tag{22}$$

with $x = (x_1\ x_2)^T$. The mass matrix M, the damping matrix C, and the stiffness matrix K are

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{23}$$

$$C = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{pmatrix} \tag{24}$$

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \tag{25}$$

The load vector is

$$F = \begin{bmatrix} \tilde{F}_1\delta(t-t_1) \\ 0 \end{bmatrix} \tag{26}$$

where $\tilde{F}_1 = 1\mathrm{N}$. A state-space vector y with length $2nN = 12$ [see remark before Eq. (16)] is introduced as

$$y = \begin{pmatrix} x & \dfrac{\mathrm{d}^{\frac{1}{3}}x}{\mathrm{d}t^{\frac{1}{3}}} & \dfrac{\mathrm{d}^{\frac{2}{3}}x}{\mathrm{d}t^{\frac{2}{3}}} & \dfrac{\mathrm{d}x}{\mathrm{d}t} & \dfrac{\mathrm{d}^{\frac{4}{3}}x}{\mathrm{d}t^{\frac{4}{3}}} & \dfrac{\mathrm{d}^{\frac{5}{3}}x}{\mathrm{d}t^{\frac{5}{3}}} \end{pmatrix}^T$$

System (22) is now rewritten as

$$\begin{pmatrix} 0 & C & 0 & 0 & 0 & M \\ C & 0 & 0 & 0 & M & 0 \\ 0 & 0 & 0 & M & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 & 0 \\ M & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \dfrac{\mathrm{d}^{\frac{1}{3}}x}{\mathrm{d}t^{\frac{1}{3}}} \\ \dfrac{\mathrm{d}^{\frac{2}{3}}x}{\mathrm{d}t^{\frac{2}{3}}} \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}^{\frac{4}{3}}x}{\mathrm{d}t^{\frac{4}{3}}} \\ \dfrac{\mathrm{d}^{\frac{5}{3}}x}{\mathrm{d}t^{\frac{5}{3}}} \\ \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} \end{pmatrix} + \begin{pmatrix} K & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & 0 & 0 & 0 & -M \\ 0 & 0 & 0 & 0 & -M & 0 \\ 0 & 0 & 0 & -M & 0 & 0 \\ 0 & 0 & -M & 0 & 0 & 0 \\ 0 & -M & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \dfrac{\mathrm{d}^{\frac{1}{3}}x}{\mathrm{d}t^{\frac{1}{3}}} \\ \dfrac{\mathrm{d}^{\frac{2}{3}}x}{\mathrm{d}t^{\frac{2}{3}}} \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}^{\frac{4}{3}}x}{\mathrm{d}t^{\frac{4}{3}}} \\ \dfrac{\mathrm{d}^{\frac{5}{3}}x}{\mathrm{d}t^{\frac{5}{3}}} \end{pmatrix} = \begin{pmatrix} F \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{27}$$

The eigenvectors $\tilde{\rho}^{(i)}$ of the system are calculated and collected in a matrix P,

$$P = \begin{bmatrix} \tilde{\rho}^{(1)} & \tilde{\rho}^{(2)} & \cdots & \tilde{\rho}^{(12)} \end{bmatrix} \tag{28}$$

Table 1 **Calculated elements b_i, with rounded figures, in diagonal matrix of Eq. (18)**

i	b_i	i	b_i
1	$1.05 - 0.692i$	7	$1.05 + 0.692i$
2	$0.806 - 0.517i$	8	$0.806 + 0.517i$
3	$-1.12i$	9	$-1.05 + 0.692i$
4	$-0.875i$	10	$-0.806 + 0.517i$
5	$-1.05 - 0.692i$	11	$1.12i$
6	$-0.806 - 0.517i$	12	$0.875i$

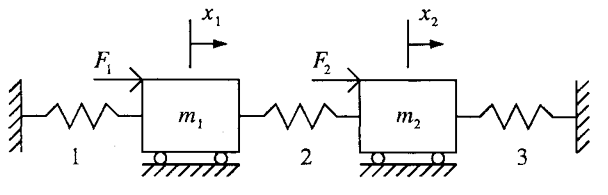

Fig. 3 Two-degree-of-freedom system. Parameters are $m_1 = 1$ kg, $m_2 = 2$ kg, $\alpha = \frac{2}{3}$, $k_2 = 1.5$ N/m, $k_1 = k_3 = 1.0$ N/m, $c_1 = c_2 = c_3 = 0.4$ Ns$^{2/3}$/m, and $l = 0$.

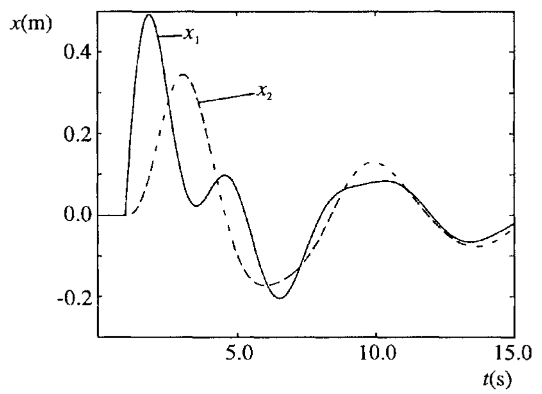

Fig. 4 Calculated displacements, as given by Eq. (30), of masses in Fig. 3, when mass m_1 is submitted to a unit impulse load at $t = t_1 = 1$ s.

Premultiplying the system of equations (27) with P^T and introducing $y = Pq$, one decouples the equations. The constants a_i in Eq. (18) are taken as unity and the constants b_i are given in Table 1.

As the load in this case is an impulse load, the integral in solution (19) is easily evaluated. The modal force $Q_i(t)$ equals a constant $\tilde{Q}_i$ multiplied by the Dirac delta function, $Q_i(t) = \tilde{Q}_i\, \delta(t - t_1)$. The modal displacements are

$$q_i(t) = \begin{cases} 0, & t < t_1 \\ \displaystyle\sum_{p=0}^{\infty} \frac{(-b_i/a_i)^p (t - t_1)^{(p-2)/3}}{\Gamma[(p+1)/3]} \frac{\tilde{Q}_i}{a_i}, & t > t_1 \end{cases} \tag{29}$$

The resulting displacements are now obtained by superposition of the modal displacements according to

$$y(t) = Pq(t) = \sum_{i=1}^{12} \tilde{\rho}^{(i)} q_i(t) \tag{30}$$

The displacements of the two masses are plotted in Fig. 4.

Eigenvector and Eigenvalue Calculation

The system of equations (16) tends to become very large when the denominator n of the order $\alpha = m/n$ of the derivative is large. When the system is rewritten from its original form (14) into the state-space form (16), no new information is added. Therefore it should be possible to calculate the desired eigenvectors from the original, smaller matrices M, C, and K rather than from the large matrices A and B.

The eigenvectors of the system (16) may be expressed by use of the eigenvalues and the eigenmodes of the original, smaller system (14),

$$\tilde{\rho}^{(i)} = \begin{bmatrix} \rho^{(i)} \\ (i\omega_i)^{1/n} \rho^{(i)} \\ \vdots \\ (i\omega_i)^{(2n+m-1)/n} \rho^{(i)} \end{bmatrix} \tag{31}$$

where $\tilde{\rho}^{(i)}$ are the $(2n + m)N \times 1$ eigenvectors of the extended system, $\rho^{(i)}$ are the $N \times 1$ eigenvectors of the original system, and $(i\omega_i)^{1/n}$ are the eigenvalues. This form is easily verified by substitution into Eq. (16). It is often convenient to denote the eigenvalues by $\lambda_i = (i\omega_i)^{1/n}$.

System (16) with $R = 0$ has $(2n + m)N$ eigenvalues $(i\omega_i)^{1/n}$. To find all of these eigenvalues, normally two different methods have to be used. By applying the Fourier transform to the system of equations (14), one obtains

$$[lM(i\omega)^{2+\alpha} + M(i\omega)^2 + C(i\omega)^\alpha + K]\hat{x} = [1 + l(i\omega)^\alpha]\hat{F} \tag{32}$$

Dividing by $[1 + l(i\omega)^\alpha]$ and letting $\hat{F} = 0$, one obtains the eigenvalue problem

$$\left[M(i\omega)^2 + \frac{C(i\omega)^\alpha + K}{1 + l(i\omega)^\alpha}\right]\hat{x} = 0 \tag{33}$$

With $\alpha = m/n$, $\lambda = (i\omega)^{1/n}$, and $\hat{x} = \rho$, this may be written

$$\left(M\lambda^{2n} + \frac{C\lambda^m + K}{1 + l\lambda^m}\right)\rho = 0 \tag{34}$$

From this eigenvalue problem the first $2nN$ eigenvalues may be found. Here the method by Abrahamsson[15] is used. That method involves Taylor's formula to rewrite the frequency-dependent stiffness matrix defined as

$$E(\omega) = M(i\omega)^2 + \frac{C(i\omega)^\alpha + K}{1 + l(i\omega)^\alpha} \tag{35}$$

Close to known approximations ω_p and ρ_p of the eigenfrequency and eigenmode, respectively, Taylor's formula yields for eigenvalue problem (33)

$$\left[E(\omega_p) + \Delta\omega \frac{\partial E(\omega_p)}{\partial \omega}\right](\rho_p + \Delta\rho) = 0 \tag{36}$$

The smallest eigenvalue $\Delta\omega$ of this linear eigenvalue problem gives a new approximation of the eigenfrequency as $\omega_{p+1} = \omega_p + \Delta\omega$.

The calculations involve evaluating the power $(i\omega)^{1/n}$ that has n different roots. By using different branches, n eigenvalues may be found, each corresponding to one branch. As starting values the positive and negative eigenfrequencies of the corresponding undamped system may be used, i.e., the system with the same matrices M and K but with l and the matrix C put to zero. With these $2N$ starting values, each yielding n eigenvalues, one may in total obtain $2nN$ eigenvalues.

A different method is needed to find the remaining mN eigenvalues. First it may be noted that if the parameter l is zero, there are only $2nN$ eigenvalues, which all may be found by the method described earlier. Bagley[16] proposed a method starting from Eq. (32). Letting $\hat{F} = 0$, one obtains the eigenvalue problem

$$(lM\lambda^{2n+m} + M\lambda^{2n} + C\lambda^m + K)\rho = 0 \tag{37}$$

where $\alpha = m/n$, $\lambda = (i\omega)^{1/n}$, and $\hat{x} = \rho$ has been used. Denoting the pth estimate of the ith eigenvalue by $\lambda_{i,p}$ and the $(p + 1)$th estimate by $\lambda_{i,p+1}$, the iterative scheme is

$$\left[M\left(l\lambda_{i,p+1}\lambda_{i,p}^{2n+m-1} + \lambda_{i,p}^{2n}\right) + C\lambda_{i,p}^m + K\right]\rho_{p+1}^{(i)} = 0 \tag{38}$$

where $\rho_{p+1}^{(i)}$ is the $(p + 1)$th estimate of the ith eigenmode. Letting

$$r = l\lambda_{i,p+1}\lambda_{i,p}^{2n+m-1} + \lambda_{i,p}^{2n} \tag{39}$$

and

$$K_d = C\lambda_{i,p}^m + K \tag{40}$$

the scheme may be written

$$K_d^{-1} M \rho_{p+1}^{(i)} = -(1/r)\rho_{p+1}^{(i)} \tag{41}$$

The eigenvalues r of this eigenvalue problem are used to calculate a new estimate of the eigenvalue $\lambda_{i,p+1}$. As there are N eigenvalues r, one may arrive at N different estimates $\lambda_{i,p+1}$. This calculation assumes that $\lambda_{i,p+1}$ is on the principal branch of $(\lambda_{i,p+1}^m)^{1/m}$. Otherwise the new estimates is calculated from the kth branch of

$$\lambda_{i,p+1} = \left[\left(\frac{r - \lambda_{i,p}^{2n}}{l\lambda_{i,p}^{2n+m-1}}\right)^m\right]^{1/m} \tag{42}$$

Discussion of Some Properties

There are some properties that may be noted to reduce the calculations. The eigenvectors $\tilde{\rho}^{(i)}$ and the eigenvalues $(i\omega_i)^{1/n}$ of system (16) satisfy

$$\left[(i\omega_i)^{1/n} A + B\right]\tilde{\rho}^{(i)} = 0 \tag{43}$$

Taking the complex conjugate (here denoted by an overbar) of this equation, one obtains

$$\left[(\overline{i\omega_i})^{1/n} A + B\right]\overline{\tilde{\rho}^{(i)}} = 0 \tag{44}$$

as the matrices A and B are real. This shows that if $(i\omega_i)^{1/n}$ is an eigenvalue, then its complex conjugate $\overline{(i\omega_i)^{1/n}}$ is also an eigenvalue, and the corresponding eigenvector is the complex conjugate $\overline{\tilde{\rho}^{(i)}}$ of the eigenvector. An exception occurs when the eigenvalue $(i\omega_i)^{1/n}$ is real. In that case Eqs. (43) and (44) are identical, and the real eigenvalues do not necessarily appear in pairs.

The constants a_i and b_i of the diagonal matrices in Eq. (18) may be written in a simpler form. Carrying out the multiplication $\tilde{\rho}^{(i)T} A \tilde{\rho}^{(i)}$ and taking into account the character of the eigenvectors, one gets

$$\begin{aligned} a_i = \tilde{\rho}^{(i)T} A \tilde{\rho}^{(i)} &= m(i\omega_i)^{(m-1)/n}\rho^{(i)T} C\rho^{(i)} \\ &+ 2n(i\omega_i)^{(2n-1)/n}\rho^{(i)T} M\rho^{(i)} \\ &+ (2n+m)(i\omega_i)^{(2n+m-1)/n}\rho^{(i)T} lM\rho^{(i)} \end{aligned} \tag{45}$$

Similarly, the constants b_i are

$$\begin{aligned} b_i = \tilde{\rho}^{(i)T} B \tilde{\rho}^{(i)} &= \rho^{(i)T} K\rho^{(i)} - (m-1)(i\omega_i)^{m/n}\rho^{(i)T} C\rho^{(i)} \\ &- (2n-1)(i\omega_i)^2\rho^{(i)T} M\rho^{(i)} \\ &- (2n+m-1)(i\omega_i)^{(2n+m)/n}\rho^{(i)T} lM\rho^{(i)} \end{aligned} \tag{46}$$

A relation between a_i and b_i may be obtained starting from the identity

$$\left[(i\omega_i)^{1/n} A + B\right]\tilde{\rho}^{(i)} = 0 \tag{47}$$

Premultiplying with $\tilde{\rho}^{(i)T}$, one obtains

$$(i\omega_i)^{1/n} a_i + b_i = 0 \tag{48}$$

i.e.,

$$-b_i/a_i = (i\omega_i)^{1/n} \tag{49}$$

It is thus seen that the constants b_i need not be explicitly calculated and that the constants a_i are easily evaluated from the original, smaller system by Eq. (45).

Numerical Example 2

A two-degree-of-freedom system like that in Fig. 3 is studied once more, but with other numerical values. Mass m_1 is now submitted to a unit step load at time t_1, $F_1 = \tilde{F}_1\Theta(t - t_1)$, where $\tilde{F}_1 = 1\text{N}$ and $\Theta(t)$ is the Heaviside function,

$$\Theta(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{50}$$

Mass m_2 carries no load. The order of the derivatives is $\alpha = \frac{1}{2}$, and the other numerical values used are given in Fig. 5. The system of equations to be solved is

$$lM\frac{d^{\frac{5}{2}}x}{dt^{\frac{5}{2}}} + M\frac{d^2x}{dt^2} + C\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} + Kx = F + l\frac{d^{\frac{1}{2}}F}{dt^{\frac{1}{2}}} \tag{51}$$

with $x = (x_1 \;\; x_2)^T$. The matrices M, C, and K are the same as in Eqs. (23–25). The $\frac{1}{2}$-order derivative of the load is found in Oldham and Spanier[11] as

$$\frac{d^{\frac{1}{2}}F_1}{dt^{\frac{1}{2}}} = \begin{cases} 0, & t < t_1 \\ \tilde{F}_1\dfrac{(t-t_1)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}, & t > t_1 \end{cases} \tag{52}$$

Introducing a state-space vector y,

$$y = \left(x \quad \frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} \quad \frac{dx}{dt} \quad \frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}} \quad \frac{d^2x}{dt^2}\right)^T$$

Table 2 Calculated eigenvalues and eigenvectors, with rounded figures, pertaining to system (53)

i	$(i\omega_i)^{\frac{1}{2}} = -b_i/a_i$	$\rho^{(i)T}$
1	$0.948 + 1.02i$	$(1;\ -0.394\ +0.00654i)$
2	$0.948 - 1.02i$	$(1;\ -0.394\ -0.00654i)$
3	$-0.951 - 0.838i$	$(1;\ -0.413\ -0.0156i)$
4	$-0.951 + 0.838i$	$(1;\ -0.413\ +0.0156i)$
5	$0.638 + 0.688i$	$(1;\ 1.26\ +0.0166i)$
6	$0.638 - 0.688i$	$(1;\ 1.26\ -0.0166i)$
7	$-0.639 - 0.569i$	$(1;\ 1.22\ -0.0266i)$
8	$-0.639 + 0.569i$	$(1;\ 1.22\ +0.0266i)$
9	-9.994	$(1;\ -0.346)$
10	-9.998	$(1;\ 1.45)$

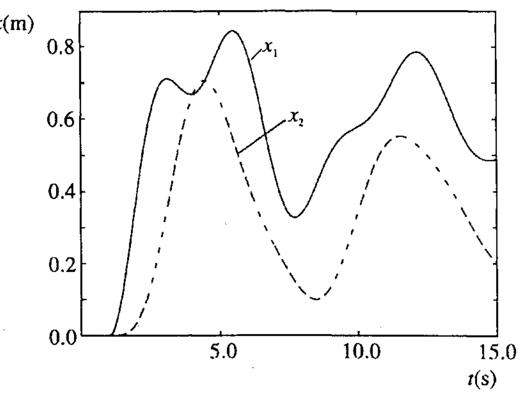

Fig. 5 Calculated displacements of masses in Fig. 3 when mass m_1 is submitted to a unit step load at $t = t_1 = 1$ s. Parameters are $m_1 = 1$ kg, $m_2 = 2$ kg, $k_1 = k_3 = 1.0$ N/m, $k_2 = 1.5$ N/m, $\alpha = \frac{1}{2}$, $c_1 = c_2 = c_3 = 0.4$ Ns$^{1/2}$/m, and $l = 0.1$ s$^{1/2}$.

[with length $(2n+m)N = 10$] the system of equations (51) may be rewritten as

$$\begin{pmatrix} C & 0 & 0 & M & lM \\ 0 & 0 & M & lM & 0 \\ 0 & M & lM & 0 & 0 \\ M & lM & 0 & 0 & 0 \\ lM & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} \\ \frac{dx}{dt} \\ \frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}} \\ \frac{d^2x}{dt^2} \\ \frac{d^{\frac{5}{2}}x}{dt^{\frac{5}{2}}} \end{pmatrix}$$

$$+ \begin{pmatrix} K & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -M & -lM \\ 0 & 0 & -M & -lM & 0 \\ 0 & -M & -lM & 0 & 0 \\ 0 & -lM & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}} \\ \frac{dx}{dt} \\ \frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}} \\ \frac{d^2x}{dt^2} \end{pmatrix}$$

$$= \begin{pmatrix} F + l\frac{d^{\frac{1}{2}}F}{dt^{\frac{1}{2}}} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{53}$$

Eight eigenvalues may be obtained by use of Eq. (36). As starting values the eigenvalues of the corresponding undamped system are used, i.e., the eigenvalues of the system

$$[M(i\omega)^2 + K]\hat{x} = 0 \tag{54}$$

The remaining two eigenvalues are obtained from scheme (41). The starting value used is the root of the equation

$$1 + l\lambda_i = 0 \tag{55}$$

i.e., $\lambda_{9,1} = -10$. All eigenvalues are listed in Table 2 together with the eigenvectors.

As expected, the eigenvalues appear in complex conjugate pairs, except for the last two eigenvalues, which are real. Also the absolute values of these two eigenvalues are larger than the absolute values of the first eight eigenvalues.

The modal displacements q_i are now obtained from Eq. (19) where the constants a_i are calculated from Eq. (45). For the first eight eigenvalues the derivative in the integral (19) may be evaluated numerically by taking enough terms of the sum in Eq. (21), but for the last two eigenvalues the sum will not converge fast enough. As the last two eigenvalues are real, the sum may be rewritten by use of the incomplete gamma function. The sum is first divided into n separate sums. Here $n = 2$ and one has

$$\sum_{p=0}^{\infty} \frac{(-b_i/a_i)^p t^{(p-1)/2}}{\Gamma[(p+1)/2]} = \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}$$

$$+ t^{-\frac{1}{2}} \left\{ \sum_{p=1,3,\ldots} \frac{\left[-(b_i/a_i)t^{\frac{1}{2}}\right]^p}{\Gamma[(p+1)/2]} + \sum_{p=2,4,\ldots} \frac{\left[-(b_i/a_i)t^{\frac{1}{2}}\right]^p}{\Gamma[(p+1)/2]} \right\}$$

$$= \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + t^{-\frac{1}{2}} \sum_{j=1}^{2} z^j \left\{ \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma[(j-1)/2 + 1 + k]} \right\} \tag{56}$$

where $z = -(b_i/a_i)t^{1/2}$ and $p = j + 2k$. According to Press et al.,[17] the sum within the last braces is almost a series development of the incomplete gamma function $\gamma[(j-1)/2, z^2]$, i.e.,

$$e^{-z^2}(z^2)^{(j-1)/2} \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma[(j-1)/2 + 1 + k]} = \gamma\left(\frac{j-1}{2}, z^2\right) \tag{57}$$

Here the incomplete gamma function is defined as

$$\gamma(a, t) = \frac{1}{\Gamma(a)} \int_0^t e^{-t} t^{a-1}\, dt, \qquad a > 0 \tag{58}$$

The sum in Eq. (56) reduces to

$$\sum_{p=0}^{\infty} \frac{(-b_i/a_i)^p t^{(p-1)/2}}{\Gamma[(p+1)/2]}$$

$$= t^{-\frac{1}{2}} \left\{ \frac{1}{\Gamma(\frac{1}{2})} + e^{z^2} \left[z\gamma(0, z^2) + |z|\gamma\left(\frac{1}{2}, z^2\right) \right] \right\} \tag{59}$$

The value of $\gamma(0, z^2)$ is taken as the limit value $\lim_{a\to 0^+} \gamma(a, z^2) = 1$ except for $z^2 = 0$. When z^2 is no longer small, the second incomplete gamma function will also yield 1 (unity), and therefore the terms within the last brackets of Eq. (59) will then cancel out, as $z = -(b_i/a_i)t^{1/2}$ is negative.

Once the sums for the derivatives have been evaluated, the integral in Eq. (19) may be calculated numerically. The resulting physical displacements are then found by superposition of the modal displacements,

$$x(t) = \sum_{i=1}^{10} \rho^{(i)} q_i(t) \tag{60}$$

The calculated displacements are shown in Fig. 5.

Concluding Remarks

The fractional derivative damping model has been studied. Small problems are seen to be easily solved in the Fourier domain. The solution may then be numerically transformed back into the time domain.

The use of a modal synthesis method to solve transient vibration problems in the time domain is explored for systems modeled with fractional derivatives. Introducing a state-space vector containing fractional derivatives of the displacement and a number of dummy equations, one expands the system of equations. The equations can then be decoupled by use of the complex eigenvectors of the system. The solutions of the resulting modal equations contain fractional derivatives of so-called Mittag–Leffler functions. Eventually the displacements are obtained by superposition of the modal displacements, weighted by the eigenvectors. It is also shown that the expanded equation matrices need not be numerically established since all information needed is obtainable from the original system. However, the present author has in some cases had convergence problems with the eigenvalue iteration method in Eq. (41) for the last eigenvalues.

To calculate the fractional derivatives of the Mittag–Leffler functions, infinite sums have to be estimated. This can here often be done by only taking into account a finite number of terms in the sums. Note that a rather large number of terms (about 300 have been used) has to be included for the sums to converge, and further studies are suggested to find a more efficient way to calculate these sums.

Appendix: Matrices A and B

The $(2n+m)N \times (2n+m)N$ [see the remark before Eq. (16)] matrices A and B in Eq. (16) are given next, where 0 denotes an $N \times N$ null matrix. The first N rows in the matrices A and B make up the system of equations (14) [these rows are marked by 1 in Eq. (A1)] and the following rows give identities in the system (16). These dummy equations are chosen to make the matrices A and B symmetric (provided that M, C, and K are symmetric). Note that if the first row and the last column of matrix A and the first row and the first column of matrix B are removed, then the remaining matrix B will be the negative of the remaining matrix A:

$$
A = \begin{array}{r}
1 \\ 2 \\ 3 \\ \vdots \\ m-1 \\ m \\ m+1 \\ \vdots \\ 2n-1 \\ 2n \\ 2n+1 \\ \vdots \\ 2n+m-1 \\ 2n+m
\end{array}
\begin{bmatrix}
0 & \cdots & 0 & C & 0 & \cdots & 0 & M & 0 & \cdots & 0 & lM \\
0 & \cdots & 0 & C & 0 & \cdots & 0 & M & 0 & \cdots & 0 & lM & 0 \\
0 & \cdots & 0 & C & 0 & \cdots & 0 & M & 0 & \cdots & 0 & lM & 0 \\
\vdots & \cdot & & \cdot & & \cdot & 0 \\
0 & \cdot & & \cdot & & \cdot \\
C & & \cdot & & \cdot & & & \vdots \\
0 & & \cdot & & \cdot & & \vdots \\
\vdots & \cdot & & \cdot \\
0 & \cdot & & \cdot & & & \cdot \\
M & & \cdot & & & & & \vdots \\
0 & & \cdot & & & & \vdots \\
\vdots & \cdot \\
0 & \cdot & 0 \\
lM & 0 & & \cdots & & \cdots & 0 & 0
\end{bmatrix}
\qquad \text{(A1)}
$$

$$
B = \begin{bmatrix}
K & 0 & & \cdots & & \cdots & & 0 \\
0 & 0 & \cdots & 0 & -C & 0 & \cdots & 0 & -M & 0 & \cdots & 0 & -lM \\
 & 0 & \cdots & 0 & -C & 0 & \cdots & 0 & -M & 0 & \cdots & 0 & -lM & 0 \\
 & \vdots & \cdot & & \cdot & & \cdot & 0 \\
 & 0 & \cdot & & \cdot & & \cdot \\
\vdots & -C & & \cdot & & \cdot \\
 & 0 & & \cdot & & \cdot & & \vdots \\
 & \vdots & \cdot & & \cdot \\
 & 0 & \cdot & & \cdot \\
\vdots & -M & & \cdot \\
 & 0 & & \cdot & & & & \vdots \\
 & \vdots & \cdot \\
 & 0 & \cdot & 0 \\
0 & -lM & 0 & & \cdots & & \cdots & 0
\end{bmatrix}
\qquad \text{(A2)}
$$

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References

[1]Nashif, A. D., Jones, D. I. G., and Henderson, J. P., *Vibration Damping*, Wiley, New York, 1985.

[2]Crandall, S. H., "The Role of Damping in Vibration Theory," *Journal of Sound and Vibration*, Vol. 11, No. 1, 1970, pp. 3–18.

[3]Gemant, A., "A Method of Analyzing Experimental Results Obtained from Elasto-Viscous Bodies," *Physics*, Vol. 7, Aug. 1936, pp. 311–317.

[4]Bagley, R. L., and Torvik, P. J., "A Generalized Derivative Model for an Elastomer Damper," *Shock and Vibration Bulletin*, No. 49, Pt. 2, 1979, pp. 135–143.

[5]Bagley, R. L., and Torvik, P. J., "Fractional Calculus—A Different Approach to the Analysis of Viscoelastically Damped Materials," *AIAA Journal*, Vol. 21, No. 5, 1983, pp. 741–748.

[6]Bagley, R. L., and Calico, R. A., "Fractional Order State Equations for the Control of Viscoelastically Damped Structures," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 2, 1991, pp. 304–311.

[7]Padovan, J., "Computational Algorithms for FE Formulations Involving Fractional Operators," *Computational Mechanics*, Vol. 2, No. 4, 1987, pp. 271–287.

[8]Makris, N., and Constantinou, M. C., "Fractional-Derivative Maxwell Model for Viscous Dampers," *Journal of Structural Engineering*, Vol. 117, No. 9, 1991, pp. 2708–2724.

[9]Gaul, L., Klein, P., and Kemple, S., "Damping Description Involving Fractional Operators," *Mechanical Systems and Signal Processing*, Vol. 5, No. 2, 1991, pp. 81–88.

[10]Enelund, M., and Olsson, P., "Damping Described by Fading Memory Models," *Proceedings of the AIAA/ASME/ASCE/AHS/ASC 36th Structures, Structural Dynamics, and Materials Conference*, AIAA, Washington, DC, 1995, pp. 207–220.

[11]Oldham, K. B., and Spanier, J., *The Fractional Calculus*, Academic, New York, 1974.

[12]Bagley, R. L., and Torvik, P. J., "On the Fractional Calculus Model of Viscoelastic Behavior," *Journal of Rheology*, Vol. 30, No. 1, 1986, pp. 133–155.

[13]Flügge, W., *Viscoelasticity*, Blaisdell, Waltham, MA, 1967.

[14]Mittag-Leffler, G., "Sur la Représentation Analytique d'une Branche Uniforme d'une Fonction Monogène," *Acta Mathematica*, Vol. 29, 1905, pp. 101–181.

[15]Abrahamsson, T., "Modal Parameter Extraction for Nonproportionally Damped Linear Structures," *International Journal of Analytical and Experimental Modal Analysis*, Vol. 3, No. 2, 1988, pp. 62–68.

[16]Bagley, R. L., "Applications of Generalized Derivatives to Viscoelasticity," Ph.D. Thesis, U.S. Air Force Inst. of Technology, 1979; also U.S. Air Force Materials Lab., TR-79-4103, Dayton, OH, 1979.

[17]Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P., *Numerical Recipes in FORTRAN, The Art of Scientific Computing*, 2nd ed., Cambridge Univ. Press, New York, 1992.